A unifying definition of synchronization for dynamical systems

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We propose a unifying definition for synchronization. By example, we show that the synchronization phenomena discussed in the dynamical systems literature fits within the framework of this definition.

Synchronization between dynamical systems has been an active research topic since the time of Huygens. It is a phenomenon of interest to fields ranging from celestial mechanics to laser physics, and from communication to neuroscience [1].

Over the last decade, a number of new types of synchronization have appeared: chaotic synchronization [2], phase synchronization [3], lag synchronization [4], and generalized synchronization [5], to mention only a few. This is in addition to the classic examples of synchronization in periodic systems [6,7]. Many of these have been experimentally observed in a single system [8]. Synchronization is often categorized on the basis of whether the coupling mechanism is uni-directional or bi-directional. Stable synchronization with uni-directional coupling has been called synchronization by an external force (for frequency synchronization) and master-slave synchronization (Pecora and Carroll [2]). (It has recently been shown that, if the synchronized systems are identical then there is no essential difference between uni-directional and bi-directional synchronization [9].)

Although there have been several attempts [10,11], no successful definition of synchronization currently exists. The definition in use is an ever increasing enumerated list. When a "new type" of synchronization arises, its name is added to the list. We believe that "definition by example" is an untidy situation which should be replaced by a single definition that encompasses all of the known examples.

In this letter we propose a unified definition which accounts for all types of synchronization between finite dimensional systems. Although we explicitly discusses synchronization between two continuous time dynamical systems, our results can be extended to N continuous time, or N discrete time systems. Therefore, our results apply to a larger class of phenomena than the one we explicitly discussed.

To construct this definition, assume that a large stationary deterministic dynamical system is divided into two sub-systems,

$$\begin{aligned} \frac{d\boldsymbol{x}}{dt} &= \boldsymbol{f}_1(\boldsymbol{x}, \boldsymbol{y}; t) \\ \frac{d\boldsymbol{y}}{dt} &= \boldsymbol{f}_2(\boldsymbol{y}, \boldsymbol{x}; t). \end{aligned} \tag{1}$$

Here, $\boldsymbol{x} \in I\!R^{d_1}$ and $\boldsymbol{y} \in I\!R^{d_2}$ are vectors that may have different dimensions. The phase space and vector field of the large system is formed (in a natural way) from the product of the two smaller phase spaces and vector fields. Examples of phenomena that can be described by Eq. (1) are ubiquitous.

Colloquially, synchronization means correlated in-time behavior between different processes. In fact, the Oxford Advanced dictionary [12], defines synchronization as "to agree in time" and "to happen at the same time". From this intuitive definition we propose that synchronization requires: (1) *Separating* the dynamics of a large dynamical system into the dynamics of sub-systems. (2) A method for *measuring properties* of the sub-systems. (3) A method for *comparing* the properties of the sub-systems. (4) A criteria for determining whether the properties *agree in time*. If they agree then the systems are synchronized. The remainder of this letter formalizes this intuitive definition of synchronization by explicitly addressing each requirement, and applying the proposed definition to examples.

We begin by separating the dynamics of the large systems into the dynamics of sub-systems. Let $\boldsymbol{\phi}(\boldsymbol{z}_0)$ denote a trajectory of the large dynamical system, given by Eq. (1), with initial condition, $\boldsymbol{z}_0 = [\boldsymbol{x}_0, \boldsymbol{y}_0] \in I\!R^{d_1} \otimes I\!R^{d_2}$. Respectively, curves $\boldsymbol{\phi}_x(\boldsymbol{z}_0)$ and $\boldsymbol{\phi}_y(\boldsymbol{z}_0)$ are obtained from this trajectory by projecting away the $\boldsymbol{y}$ and $\boldsymbol{x}$ components. We say that $\boldsymbol{\phi}_x(\boldsymbol{z}_0)$ and $\boldsymbol{\phi}_y(\boldsymbol{z}_0)$ are "trajectories" of the first and second sub-systems of Eq. (1). In this context we have *separated* the trajectories $\boldsymbol{\phi}_x(\boldsymbol{z}_0)$ and $\boldsymbol{\phi}_y(\boldsymbol{z}_0)$ from $\boldsymbol{\phi}(\boldsymbol{z}_0)$, rather than constructing $\boldsymbol{\phi}(\boldsymbol{z}_0)$ from $\boldsymbol{\phi}_x(\boldsymbol{z}_0)$ and $\boldsymbol{\phi}_y(\boldsymbol{z}_0)$.

To discuss *measuring properties* of the sub-systems let $\mathcal{X}$ denote the space of all trajectories of the first sub-system, and consider a mapping $\boldsymbol{g}_x : \mathcal{X} \otimes I\!R \to I\!R^k$. The first $I\!R$ represents time, and is included so that $\boldsymbol{g}_x$ can make explicit reference to time. We say that the mapping, $\boldsymbol{g}_x$, is a *property* of the first sub-system. The image of $[\boldsymbol{\phi}_x(\boldsymbol{z}_0), t] \in \mathcal{X} \otimes I\!R$ under the mapping $\boldsymbol{g}_x$ is the result of *measuring the property* of the first sub-system. It will be denoted by $\boldsymbol{g}(\boldsymbol{x}) \in I\!R^k$. Similar definitions

can be made for the second sub-system. The following examples make these notions less abstract.

For synchronization, a property of a sub-system that is often of interest is frequency. Measuring the property means calculating a numerical value for the frequency. Hence, $\omega_x = \boldsymbol{g}(\boldsymbol{x})$. Other properties of interest are the coordinates of a sub-system at time t. Measuring the properties means determining numerical values for the coordinates. Hence, $\boldsymbol{x}(t) = \boldsymbol{g}(\boldsymbol{x})$. Experimentally, $\boldsymbol{g}_x$ is the quantity being measured, and $\boldsymbol{g}(\boldsymbol{x})$ is the value of the measurement. These examples show that a property can be a long time average, or a quantity whose value depends implicitly on time. Furthermore, the dimension of the measurement, k, can take on different values depending on the property being measured. Notice that it is reasonable to say $\boldsymbol{g}_x$ is a property of the first sub-system because $\boldsymbol{g}(\boldsymbol{x})$ is obtained without explicitly referring to any other sub-system.

Finally, we discuss the notions of *comparing* the properties, and determining when they *agree in time*. We say the function $\boldsymbol{h} : \mathbb{R}^k \otimes \mathbb{R}^k \rightarrow \mathbb{R}^k$ *compares* the measured properties of the two sub-systems, and the two measurements *agree in time* if and only if $\boldsymbol{h}[\boldsymbol{g}(\boldsymbol{x}), \boldsymbol{g}(\boldsymbol{y})] = \boldsymbol{0}$. Below, a norm is used to determine this last requirement.

With these preliminaries in place, we offer the following definition for synchronization:

Definition *The sub-systems in Eq. (1) are synchronized on the trajectory $\boldsymbol{\phi}(\boldsymbol{z}_0)$, with respect to the properties, $\boldsymbol{g}_x$ and $\boldsymbol{g}_y$, if there is a time independent function $\boldsymbol{h}$: $\mathbb{R}^k \otimes \mathbb{R}^k \rightarrow \mathbb{R}^k$ such that*

$$\|\boldsymbol{h}[\boldsymbol{g}(\boldsymbol{x}), \boldsymbol{g}(\boldsymbol{y})]\| = 0,$$

where $\| \bullet \|$ is some norm.

A subsequent definition removes details of initial conditions and trjectories: The sub-systems are synchronized with respect to the properties $\boldsymbol{g}_x$ and $\boldsymbol{g}_y$ if the previous definition holds on *all* trajectories. The subsequent definition is what many papers in the literature call synchronization [1]. However, as we and others have shown, synchronization depends strongly on the trajectory [1,13]. Therefore, the trajectory dependence in the first definition can not be ignored.

We claim that this definition naturally follows from the intuitive definition of synchronization, and that it encompasses all of the interesting examples found in the literature. A strength of this definition is that the properties and comparison function are not specified, *a priori*. As shown below, different applications *require* different properties and comparison functions, and those that are suitable for one application are often completely unsuitable for another. For example, the following comparison functions all appear in the literature

$$\boldsymbol{h}[\boldsymbol{g}(\boldsymbol{x}), \boldsymbol{g}(\boldsymbol{y})] \equiv \boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{y}) \tag{2}$$

$$\boldsymbol{h}[\boldsymbol{g}(\boldsymbol{x}), \boldsymbol{g}(\boldsymbol{y})] \equiv \lim_{t \to \infty} [\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{y})], \tag{3}$$

$$\boldsymbol{h}[\boldsymbol{g}(\boldsymbol{x}), \boldsymbol{g}(\boldsymbol{y})] \equiv \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} [\boldsymbol{g}(\boldsymbol{x}(s)) - \boldsymbol{g}(\boldsymbol{y}(s))] ds. \tag{4}$$

Similar breadth occurs with properties. The same two sub-systems may be synchronized with respect to some properties, yet not synchronized with respect to other properties. A lack of breadth is the Achilles heel of previous definitions. Typically, they fail because they, *a priori*, specify properties and/or comparison functions that must be applied to *all* types of synchronization. Therefore, it is easy to find examples where the selected property and/or comparison function is inappropriate.

Some may be concerned that the definition is too general. We argue, via analogy, that generality is also strength. Perhaps the most useful concept in theoretical physics is a vector space. The definition of a vector space is as general as the one proposed for synchronization [14]. The definition does not specify what constitutes a "vector" or the operation "+". Thus, a range of things from matrices to Fourier series to bras and kets are vectors in their respective vector spaces. The definition only insists that the set of "vectors" obey a specific series of abstract rules. If a set obeys these rules then it is a vector space, and the considerable power one obtains from that knowledge can be employed. (Group theory is another example of an extremely useful concept in physics whose definition is abstract.) Our definition, gives an explicit list of four tasks and a condition that must be satisfied for synchronization. Like the definition of a vector space (or a group) it provides a structural framework that can be used for subsequent research. We submit that this structure is an improvement over the current situation of failed definitions and enumerated lists.

The remainder of this letter demonstrates the utility of the definition by discussing well-known examples.

Frequency Synchronization

Sub-system properties used in frequency synchronization are frequencies. If the trajectory is mostly rotation about an axis then the measured frequencies ($\omega_x = \boldsymbol{g}(\boldsymbol{x})$ and $\omega_y = \boldsymbol{g}(\boldsymbol{y})$) are located at power spectra peaks associated with the average rotation of the signal. Examples of such dynamics include, periodic motion, systems with phase coherent chaotic attractors (like Rössler [15]), or systems with Šilnikov dynamics [16]. For these examples, phase modulation contributes weakly to the dynamics, and chaos (if it exists) results mainly from amplitude modulation.

The measurement function is typically $\boldsymbol{h}[\boldsymbol{g}(\boldsymbol{x}), \boldsymbol{g}(\boldsymbol{y})] \equiv n_x\omega_x - n_y\omega_y$ (where n_x and n_y are integers). Synchronization implies that the frequencies of the sub-systems are commensurate

$$n_x\omega_x - n_y\omega_y = 0. \tag{5}$$

Many text books discuss frequency synchronization for periodic systems [7].

Frequency synchronization between coupled chaotic systems with bi-stable attractors has also been examined [17]. (The Lorenz and double scroll attractors are examples of bi-stable attractors.) In Ref. [17], the properties are the average frequency of switching between the two lobes of the attractors, and frequency synchronization on a trajectory occurs if Eq. (5) is satisfied. Our definition works for this example. Many other definitions fail because, either the definition of phase is ambiguous [18], and/or $\|\boldsymbol{x}-\boldsymbol{y}\|$ need not remain small [10].

Another example is frequency synchronization between a chaotic bi-stable attractor and a periodic system. The communication method of Hayes *et. al.* [19] labels the lobes of the attractor as 0 or 1, and uses small control signals to produce a trajectory which encodes the message. The obvious choice for the sampling rate of the receiver is the mean switching frequency of the chaotic system. Therefore, a periodic receiver with frequency ω_y is synchronized to a chaotic transmitter with switching frequency ω_x if $\omega_x = \omega_y$. This type of synchronization fits into our definition, but does not seem to fit into any previous definition.

Frequency synchronization compares properties that are long time averages of the trajectory. Therefore, it is a loose restriction on the dynamics of the sub-systems. In particular, it does not restrict the instantaneous values of the coordinates $\boldsymbol{x}$ and $\boldsymbol{y}$. All remaining examples compare properties whose measured values depend implicitly on time.

Phase Synchronization

Phase synchronization involves sub-system properties called "phases". If the dynamics is chaotic and phase coherent then one can introduce cylindrical coordinates, and unambiguously define the phase as the angle coordinate, $\phi(t)$. However, other applications define the phase via a Hilbert transform, in which case the phase may not be uniquely defined on the sub-system [3]. Also, there are examples where the measured phase, $\boldsymbol{\phi}(t)$, is a vector obtained from a trajectory using none of the previous methods [18].

If the measured properties are given by $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{\phi}_x(t)$ and $\boldsymbol{g}(\boldsymbol{y}) = \boldsymbol{\phi}_y(t)$ then the most common comparison function is [3,4,20],

$$\boldsymbol{h}[\boldsymbol{g}(\boldsymbol{x}), \boldsymbol{g}(\boldsymbol{y})] = \boldsymbol{U}\left[\boldsymbol{\epsilon}, (\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{y}))\right]. \tag{6}$$

Here, $\boldsymbol{U}(\boldsymbol{u}, \boldsymbol{v})$ is a vector with α-th component $U_\alpha(\boldsymbol{u}, \boldsymbol{v}) = \Theta\left[u_\alpha - |v_\alpha|\right]$, and Θ is the unit step function. Equation (6) says that synchronization means $|\phi_{x\alpha} - \phi_{y\alpha}| < \epsilon_\alpha$, so $\boldsymbol{\epsilon}$ is the maximum tolerable separation between the components of the phase. The value of $\|\boldsymbol{\epsilon}\|$ is usually small, but can not be set *a priori* because its size is application dependent [20]. If "phase slips" occur then a comparison function using a time average like that of Eq. (4) is necessary [3].

Phase synchronization only compares the phase variables. In the synchronous state the phases are locked, but the amplitudes can remain chaotic and relatively uncorrelated. Our definition includes phase synchronization. In contrast, definitions which focus on $\|\boldsymbol{x}-\boldsymbol{y}\|$ [10] fail because phase synchronization does not restrict amplitudes. Likewise, any definition that forces one to use a specific type of phase will fail because phase is not uniquely defined.

Identical Synchronization

This is the most frequently discussed form of synchronization within the nonlinear dynamics community [1]. Here the sub-systems are identical, and the properties are the phase space variables, $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{x}(t)$, and $\boldsymbol{g}(\boldsymbol{y}) = \boldsymbol{y}(t)$. Most discussions in the literature use Eq. (3) as the comparison function [1,10].

However, if the dynamics of the sub-systems are chaotic then bursts (sudden loss and recovery of synchronous motion caused by unstable periodic orbits within the attractor) occur on chaotic trajectories for some forms of coupling [13]. Applications which can not tolerate bursts demand high quality synchronization, where Eqs. (6) is used as the comparison function [20]. If bursts are tolerable then a hybrid of Eqs. (4) and (6) can be used.

An engineering application called "dead-beat" synchronization (only possible in discrete time dynamical systems) uses Eq. (2) as the comparison function [21]. This type of synchronization is often used to describe systems whose measured properties are restricted to a finite symbolic alphabet [22].

In their seminal paper, Afraimovich, Verichev, and Rabinovich [23] generalized identical synchronization in two different ways.

Lag Synchronization

Two sub-systems are lag synchronized if their measured properties lag each other by a fixed amount of time, τ. A trivial example is when the measured properties are $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{x}(t)$ and $\boldsymbol{g}(\boldsymbol{y}) = \boldsymbol{y}(t+\tau)$, and the comparison function is Eq. (2). For this example, the second sub-system follows the same trajectory as the first sub-system, but is τ units of time behind.

A nontrivial example is Ref. [4]. In this paper the measured properties are $\boldsymbol{g}(\boldsymbol{x}) = x_1(t)$ and $\boldsymbol{g}(\boldsymbol{y}) = y_1(t+\tau)$ (the first components of $\boldsymbol{x}$ and $\boldsymbol{y}$). The comparison function is $\boldsymbol{h}[\boldsymbol{g}(\boldsymbol{x}), \boldsymbol{g}(\boldsymbol{y})] = K\left\langle [\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{y})]^2 \right\rangle$, where K is a constant and $\langle \bullet \rangle$ is a time average. For this example, the sub-systems are *not* identical and $S^2(\tau) \equiv \|\boldsymbol{h}\| = 0$ for a *non-zero* value of τ.

Lag synchronization also occurs if, instead of a constant value of τ, one uses $\boldsymbol{g}(\boldsymbol{y}) = \boldsymbol{y}[T(t)]$, with $T : I\!R \to I\!R$ a homeomorphism with $\lim_{t\to\infty} \frac{T(t)}{t} = 1$ [23].

The second generalization in Ref. [23] follows from the observation: If the sub-system are identical then the set $\boldsymbol{x} = \boldsymbol{y}$ defines an invariant manifold in the phase space of the large system.

Generalized Synchronization

The literature is not consistent when discussing generalized synchronization. Most papers say that generalized synchronization occurs if the measured properties are $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{x}$, $\boldsymbol{g}(\boldsymbol{y}) = \boldsymbol{y}$, and the comparison function, $\boldsymbol{h}$, is given by

$$\boldsymbol{h}[\boldsymbol{g}(\boldsymbol{x}), \boldsymbol{g}(\boldsymbol{y})] = \boldsymbol{H}[\boldsymbol{g}(\boldsymbol{x})] - \boldsymbol{g}(\boldsymbol{y}), \tag{7}$$

where $\boldsymbol{H}$ is a smooth, invertible, time independent function [11]. Roughly speaking, the sub-systems are generally synchronized if $\boldsymbol{y}(t) = \boldsymbol{H}[\boldsymbol{x}(t)]$. The equation, $\boldsymbol{y} = \boldsymbol{H}(\boldsymbol{x})$, defines an invariant manifold in the phase space of the large system, and one can determine the state of one sub-systems from the state of the other sub-system [11].

However, Rulkov *et. al.* [5] examined an example where the sub-systems have the same functional form but different parameter values. Their numerical and experimental evidence indicates that it is possible to have stable frequency synchronization on a trajectory and not have generalized synchronization in the sense discussed above. For their example, one sub-systems oscillated twice for every oscillation of the other sub-system (i.e., $\omega_1/\omega_2 = 2$). This implies that it is impossible to construct a smooth invertible mapping $\boldsymbol{y} = \boldsymbol{H}(\boldsymbol{x})$. Therefore, the definitions in Ref. [11] fail.

This example illustrates that the definition of generalized synchronization needs to include $\boldsymbol{H}$'s with a finite (or perhaps countable) number of branches. Similar conclusions arise from Ref. [24]. Because we only require the functional reationship, $\boldsymbol{h}[\boldsymbol{g}(\boldsymbol{x}), \boldsymbol{g}(\boldsymbol{y})] = \boldsymbol{0}$, between the properties of the sub-system, both of these examples are naturally contained within our definition of synchronization. In particular, we do not insist that this relationship have the form $\boldsymbol{g}(\boldsymbol{y}) = \boldsymbol{H}[\boldsymbol{g}(\boldsymbol{x})]$. Therefore, we argue that the definition we have proposed is a more natural definition for generalized synchronization.

In conclusion, we have describe four tasks that are required for synchronization. Based on this, we proposed a unified definition of synchronization between finite dimensional dynamical systems. We claim that this definition encompasses all examples of synchronization discussed in the literature, and that it offers a common language and framework that can be used to discuss different types of synchronization.

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[1] *Chaos* **7** (1997); *IEEE Trans. Circuits and Systems, part I* **44** (1997), and references therein.
[2] L. M. Pecora, and T. L. Carroll, *Phys. Rev. Letts.* **64**, 821 (1990).
[3] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, *Phys. Rev. Letts.* **76**, 1804 (1996).
[4] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, *Phys. Rev. Letts.* **78**, 4193 (1997).
[5] N. F. Rulkov, M. M. Sushchik, L. S. Tsimring, and H. D. I. Abarbanel, *Phys. Rev.* **51E**, 980 (1995); N. F. Rulkov and M. M. Sushchik, *Phys. Lett.* **214A**, 145 (1996).
[6] I. I. Blekhman, P.S. Landa, and M. G. Rosenblum, *Appl. Mech. Rev.* **48**, 733 (1995).
[7] See the discussion of Arnold tongues in E. Ott, *Chaos in dynamical systems* (Cambridge University Press, New York, NY, 1994).
[8] D. Y. Tang, R. Dykstra, M. W. Hamilton, and H. R. Heckenberg, *Chaos* **8**, 697 (1998).
[9] K. Josic, *Phys. Rev. Letts.* **80**, 3054 (1998).
[10] P. F. Curran and L. O. Chua, *Int. J. Bif. and Chaos* **7**, 1375 (1997); and papers in Ref. [1].
[11] L. Kocarev and U. Parlitz, *Phys. Rev. Letts.* **76**, 1816 (1996); U. Parlitz, L. Junge, and L. Kocarev, *Phys. Rev. Letts.* **79**, 3158 (1997); B. R. Hunt, E. Ott, and J. A. Yorke, *Phys. Rev.* **55E**, 4029 (1997); R. Brown, *Phys. Rev. Letts.* **81**, 4835 (1998).
[12] A. S. Hornby, *Oxford Advanced Dictionary*, (Oxford University Press, 1974).
[13] P. Ashwin, J. Buescu, and I. Stewart *Phys. Lett.* **193A**, 126 (1994); P. Ashwin, J. Buescu, and I. Stewart *Nonlinearity* **9**, 703 (1996); R. Brown and N. F. Rulkov, *Phys. Rev. Letts.* **78**, 4189 (1997); R. Brown and N. Rulkov, *Chaos* **7**, 395 (1997).
[14] F. W. Byron and R. W. Fuller, *Mathematics of Classical and Quantum Physics*, (Addison-Wesley, 1969), p 88; J. Mathews and R. L. Walker, *Mathematical Methods of Physics*, (W. A. Benjamin, Inc., 1964), p 140.
[15] J. D. Farmer, *Phys. Rev. Letts.* **47**, 179 (1981); J. Crutchfield, D. Farmer, N. Packard, R. Shaw, G. Jones, and R. J. Donnelly, *Phys. Lett.* **76A**, 1 (1980); E. F. Stone, *Phys. Lett.* **163A**, 367 (1992) .
[16] K. Judd and A. Mees, *Physica* **82D**, 426 (1995).
[17] V. S. Anishchenko, A. N. Silchenko, and I. A. Khovanov, *Phys. Rev.* **57E**, 316 (1998).
[18] T. Yalcinkaya and Y-C Lai, *Phys. Rev. Letts.* **79**, 3885 (1997).
[19] S. Hayes, C. Grebogi, and E. Ott, *Phys. Rev. Letts* **70**, 3031 (1993).
[20] D. J. Gauthier, and J. C. Bienfang, *Phys. Rev. Letts.* **77**, 1751 (1996).
[21] A. De Angeli, R. Genesio, and A. Tesi, *IEEE Trans. Circuits and Systems, part I* **42**, 54 (1995).
[22] W. E. Savage, *Bell Syst. Tech. J.* **46**, 449
[23] V. S. Afraimovich, N. N. Verichev, and M. I. Rabinovich, *Radiophys. Quantum Elect.* **29**, 747 (1986).
[24] K. Pyragas, *Phys. Rev.* **54E**, R4508 (1996).